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SELECTION, SOCIALIZATION AND STABLE NETWORKS

Murat Ali ÇENGELCİ[1]

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Abstract

When individuals make decisions or choose behaviors, they are affected by the decisions or behaviors of their friends. In turn, individuals often choose their friend based on similarities in these decisions and behaviors. Kandel (1978) provide empirical evidence that these aspects are present in friendship networks and refers them as socialization and selection, respectively. In this paper, we consider a model of friendship network which reflects both aspects. To capture socialization, we consider a network game where agents with different types choose an action on a certain behavior to coordinate their action both with their type and actions of friends in network. Agent, then, decide their friends based on the utility of network game under unique Nash equilibrium. We show certain class of networks are pairwise stable.

Keywords: Selection, Socialization, Social Network Theory

Jel Classification: C72, D85

Article Type: Research Article

ÖN SEÇİM, SOSYALLEŞME VE STABİL AĞLAR

Öz

İnsanlar tercihler yaparken veya davranışlarına karar verirken arkadaşlarının karar ve tercihlerinden etkilenirler. Öte yandan insanlar arkadaşlık kararlarını verirken de arkadaşlarının tercih ve kararlarından etkilenirler. Kandel (1978) araştırmasında bu iki yönlü etkinin var olduğunu ampirik olarak göstermiş ve bu etkileri sosyalleşme ve ön seçim olarak adlandırmıştır. Bu makalede biz insanların arkadaşlıklarına karar verdikleri bir sosyal ağ modeli öneriyoruz. Sosyalleşme etkilerini çalışabilmek için bireylerin tercih ve kararlarını arkadaşları ile koordine ettiği bir ağ oyunu modelliyoruz. Ön seçim etkilerini çalışabilmek içinse bireylerin ağ oyununda tek olduğunu

[1] Dr., İstanbul Medeniyet University, Siyasal Bilgiler Fakültesi, İktisat Bölümü, cengelci@cengelci.com, https://orcid.org/0009-0000-3828-0827

gösterdiğimiz Nash Dengesine dayalı olan fayda fonksiyonuna göre arkadaşlık kararlarını verdiklerini varsayıyoruz. Belli bazı sosyal ağların da stabil ağ olduğunu gösteriyoruz.

Anahtar Kelimeler: Ön Seçim, Sosyalleşme, Sosyal Ağlar Teorisi

Jel Sınıflandırması: C72, D85

Makale Türü: Araştırma Makalesi

I. GİRİŞ

The individuals who are connected via a social networks affect each other's decisions and behaviors[2] and in turn individual characteristics determine which individuals we choose to be connected with[3]. In other words, individual characteristics form the basis of social relations and the nature of these social ties alter these characteristics through social influence and peer effects. Kandel (1978) provides evidence on this circle of influence particularly for friendship networks that exhibit homophily[4].

Homophily refers to observed similarities in behaviors and personal characteristics of individuals who affiliate with each other. According to Kandel (1978), observed level of homophily in a network at a particular time can be explained by two processes: selection leading to assortative matching of pairs by deliberate choice of friendship based on similarities on some behavior or characteristics and socialization leading to adjustments in behaviors to mitigate the discrepancy between own behavior and that of friends. Kandel (1978) shows that adolescents coordinate their choice of friends and their behaviors, such as frequency of marijuana or involvement in delinquent activities, in order to maximize concordance in behaviors between friends and in case of any discordance, the adolescent either break off friendship or update own behavior to match his friend's behavior.

In this paper, we consider a model of friendship network that exhibits both selection and socialization. To capture socialization, we consider finite set of agents who are connected via a social network. Agents have types which are interpreted as proclivity towards a certain behavior. Agents, then, choose an action which is interpreted as the actual action on the

[2] The list of decisions or behaviors that are affected by network structure includes which product to buy Dubey, Garg, and Meyer (2012), learning Golub and Jackson (2012a), Golub and Jackson (2012b), Golub and Jackson (2010), criminal activity Glaeser et al. (1996), political participation Knocke (1990), educational attainment Cohen (1977), Patacchini et al. (2011), Calvó-Armengolet al. (2009) and drug use McCabe et al. (2005).

[3] For example, individuals decide to form, maintain or sever a relationship based on mutual benefit or cost arises from various networks (such as, trading relationships, professional collaborations, political alliances) or individuals simply seek similarities in certain characteristics with those they are associated.

[4] See McPherson, Smith-Lovin, and Cook (2001) for a comprehensive survey regarding the vast empirical evidence on homophily, a term proposed by Lazarsfeld and Merton (1954).

behavior given a network and friends' actions. Examples of types and actions can be parents' education level and an adolescent's educational attainment, crime level in an agent's neighborhood or family members and level criminal activities the agent involved. When choosing an action, agents want to minimize the difference between their own action and friend's action as well as the difference between their action and type. The utility of an agent is weighted sum of two coordination benefits which are evaluated by the same functional form and normalized to 1 in case there is no difference. Coordination benefits decrease quadratically as differences mentioned above increase and a parameter, we call tolerance rate, determines the magnitude of decreases in coordination benefits. In other words, network structure determines strategic interaction among agents. Therefore, network games[5] arise as a natural candidate to study strategic interaction due to socialization. In a network game, payoff of an individual depends on actions of his friends and on the overall network structure because of the possible feedback effects.

One particular example of network games which is closely related with socialization part of this paper is Ballester et al. (2006)[6] which considers a network game with linear-quadratic payoffs exhibiting strategic complementarity between friends' effort levels and own effort level. They show that there is a unique Nash equilibrium when complementarities are small compared to cost and benefit of exerting effort. They also show that the Nash equilibrium action of an agent is proportional to his Bonacich-Katz centrality score which embodies feedback effects of underlying network[7]. The agents are homogenous in terms of observable characteristics but they comment about heterogeneity in a remark and how their results change. But the heterogeneity among agents is in terms of a parameter in the utility function which affects how an agent evaluates the cost and benefit of exerting effort independent of network effects on effort levels[8]. However, in our model, heterogeneity among agents is modelled as intrinsic characteristics of agents which yields to meaningful comparisons of an agents type and action; whether an agent takes a higher or lower action than her type. In particular, an agent's action is equal to his type in the absence of a network

[5] Network games studied extensively. See Jackson and Zenou (2012) for a recent review on network games. There are two class of network games depending on nature of strategic interaction; where actions are strategic complement or substitute. Examples of network games with strategic substitutes are local public good provision Bramoullé and Kranton (2007), information gathering (sharing of costly information). Examples of network games with strategic complementarity are adoption of new technologies (See Morris (2000)), education decisions and criminal activity. For an example of game with mixed strategic interaction, see Goyal and Moraga-González (2001) analyzing R&D and competition.

[6] It is quite influential paper. There are various papers based on their model. For example, Helsley and Zenou (2011) shows that the Nash equilibrium is not efficient. Calvó-Armengoi and Zenou (2004) and Ballester et al. (2010) offers a policy proposal about choosing the key criminal so that removal of that criminal reduce overall criminal activity most. See Calvó-Armengol et al. (2009) and Ballester et al. (2011) for applications to education, Koenig et al. (2012) for R&D partnership between competing firms.

[7] See section 2.3 for details of Bonacich-Katz centrality measure.

[8] In empirical studies, heterogeneity in characteristics includes average of friend's characteristic. See Calvó-Armengol et al. (2009) for education and Patacchini and Zenou (2012) for crime.

and his action differs from his type due to socialization. In our model, Nash equilibrium actions are proportional to the weighted Bonacich-Katz centrality score and it is unique as long as it is costly to choose an action different than his type.

Selection states that individuals in friendship networks choose their friends based on similarities in behaviors. Friendship relations are reciprocal, that is, forming a link requires consent of both agents involved in the link, but one can end the friendship without consent of other. Pairwise stability captures this reciprocity in friendship relations: A network is said to be pairwise stable when no agent wants to sever a link or no two agents want to form a link. Since Nash equilibrium of network game is unique, we set utility of a network to be equal to utility of agent under equilibrium actions of that network game. This allows agents to decide to terminate a link or add a new one by comparing utilities under different networks. Strategic network formation models usually assume the utility that an agent gets from a network without being explicit about where these utilities are derived from[9]. By establishing a link between network game and network formation, we provide that underlying process for the utility of networks: utility of network is derived from the underlying network game. We show that empty network is pairwise stable when tolerance rate is small enough. For other values of tolerance rate, finding pairwise stable networks becomes quite complicated because of feedback effects of network structure on an agent's action. We, however, show two classes of networks to be pairwise stable. First class of networks, we call dyadic networks, have components containing at most two agents whose types are closest to each other. Second class of network can contain three agent whose types are closest to each other in a component. For the special case of n=5, we show all pairwise stable network and parameter for which they are pairwise stable. As tolerance rate and peer effect increase, certain class of interval networks will be pairwise stable. Agents first form links with agents closest to them in type due to selection process. But once the relationships form, due to socialization process, agents in same component coordinate their behavior[10] and exhibit similar behavior. This also increase the incentive to form further relationship within a component turning it to a clique. Although adjacent cliques or components coordinate their behaviors within themselves and average behavior differs from each other, for high enough parameter values, agents form relationships

[9] For example, Bloch and Jackson (2007) assumes that agents have a constant benefit from friend-ship which decays with the distance in network and agents incur a constant cost of maintaining a relationship with their direct friends but there is no underlying process or mechanism that deter-mines those benefits and costs. They show pairwise stable networks for different values of cost and benefit and the rate benefits decays. Jackson and Wolinsky (1996) consider negative externalities due to links.

[10] Due to feedback effects, even if two agents in same component are not directly connected, the behavior became similar.

with adjacent components and eventually complete network, where every agent connected to all others, will be pairwise stable.

Both aspects of social networks are studied in the literature separately: socialization in the context of network games and selection as a part of strategic network formation literature. These studies provide insights on each of these in separation from other. However, they remain silent on possible interdependencies or what kind of network patterns can emerge when two processes are active at the same time. There are few attempts to bring these two literatures together in a unified framework. Koenig et al. (2011) is the closest to ours and differs from this paper mainly in the way they treat of differences among individuals. As they follow Ballester et al. (2006), they abstract from ex-ante heterogeneities among individuals in terms of their personal characteristics and agent's positions in network is the only source of heterogeneity whereas we bring these natural heterogeneities through types. They consider a dynamic network formation process starting from empty network. We, however, consider a static network formation by employing pairwise stability as our solution concept which reflects a characteristics of friendship networks: a relationship demands approval of both agents involved in relationship while an agents can end a relationship without approval of other agent.

In section II, we introduce social networks and Bonacich-Katz centrality measure. In section III, we model socialization as network game and show the Nash equilibrium of the game as a function of generalized Bonacich-Katz centrality measure. In section IV, we show pairwise stable networks for various values of parameters of model. Finally, we conclude in last section. All proofs are in appendix.

II. NETWORKS

A network is a pair (N, G) where N is a finite set of agents and $G \in \{0,1\}^{\{n\times n\}}$ is adjacency matrix representing the links between agents. $g_{ij} = 1$ is interpreted as agents i and j are friends or linked[11]. Throughout the paper, we assume N is fixed, so we simply denote a network as G. We assume that links are formed based on mutual consent, hence the network matrix G is assumed to be symmetric. Let $\mathcal{G}$ be the set of all symmetric networks.

[11]We follow the convention of setting $g_{ij} = 0$ for all $i \in N$.

With a slight abuse of notation, we write $ij \in G$ whenever $g_{ij} = 1$ and $G + ij$ has the natural interpretation that $G + ij$ is the network obtained by adding a link between i and j while $G - ij$ is the network that is same with G except the link between i and j is severed[12].

Let $N_i(G) = \{j \in N: g_{ij} = 1\}$ represent the set of neighbors of i and $d_i(G) = \#\, N_i(G)$ represent the degree of agent i while $d(G) = \big(d_1(G), \dots, d_n(G)\big)$ is the degree vector and $D(G) = diag\big(d(G)\big)$ is the diagonal degree matrix with diagonal equals to degree vector and off-diagonals are all zero. We simply write d and D for degree vector and diagonal degree matrix when the network G is clear in the context.

Paths, components and cliques

We say that there is a path between i and j of length K if there is a sequence of agents $\{i_0, \dots, i_K\}$ such that $i_0 = i$ and $i_K = j$ and $g_{i_k i_{k+1}} = 1$ for $k = 0, \dots, K-1$. Notice that a path between i and j is not necessarily shortest path and it is allowed to have loops[13]. We say agents i and j are *directly connected* if $ij \in G$, *indirectly connected* if $ij \notin G$ but there is path connecting i and j and otherwise, *disconnected* if there is not path connecting two agents. A network G is said to be connected if there is a path between every two agents at some finite length.

Given a set $S \subset N$ and network G, we define subnetwork $G|_S$ as the network G restricted to set S. More formally,

$$[G|_S]_{ij} = \begin{cases} 1 & if\ i,j \in S, g_{ij} = 1 \\ 0 & otherwise \end{cases}$$

In matrix notation, $G|_S$ is a principal submatrix of G.

A *component* of a network is a partition $C(G) = \{S_1, \dots, S_K\}$ such that (i) all sub-networks are connected; $G|S_K$ for all $k = 1, \dots, K$ and (ii) there is no path between any two agents belonging different cells. Component is the set of maximal connected sets and for the network in figure 1, the component is $C(G) = \{1,2,3\}, \{4,5,6,7,8\}, \{9\}$. Let $C_i(G) = \{S \in C(G): i \in S\}$ be the component that agent i belongs.

[12] A network is a graph which is a pair $(V(G), E(G))$ where $V(G)$ is the set of agents (nodes, vertices) in G and $E(G)$ is the list of links (edges) between agents. The list of links $E(G) \subseteq \{K \subset N: |K| = 2\}$ is a subset of unordered pairs of agents of size 2.

[13] There is no restriction that the elements of sequence $\{i_0, \dots, i_K\}$ are distinct, hence the definition allows loops.

A *multiset* is a family $\{Q_1, \dots, Q_K\}$ of subsets of N, which allows the possibility that $Q_i \cap Q_j \neq \emptyset$ but $\cup_i Q_i = N$. A *clique* is a subnetwork $G|_Q$ which is complete subnetwork and a *maxclique* is a clique that is not contained in another clique. In the network in figure 1, for example, the clique {4,5,6} is not a maxclique since it is contained in another clique {4,5,6,7} which is a maxclique and it has a non-empty intersection with maxclique {7,8}. A *clique-cover* a network is a multiset $\{Q_1, \dots, Q_K\}$ where each Q_k is a clique. There is always a clique cover and usually it is not unique. A *maxclique cover* of a network G is a multiset $Cl(G) = \{Q_1, \dots, Q_K\}$ with each Q_k is a maxclique. A maxclique cover exists for all networks, moreover it is unique. The network in figure 1 has a maxclique cover $Cl(G) = \{1,2\}, \{2,3\}, \{4,5,6,7\}, \{7,8\}, \{9\}$.

The difference between a component and a maxclique is that subnetwork on a component can be any subnetwork as long as it connected while subnetwork on maxclique has to be a complete subnetwork. For example, {1, 2, 3} is a component but not a maxclique.

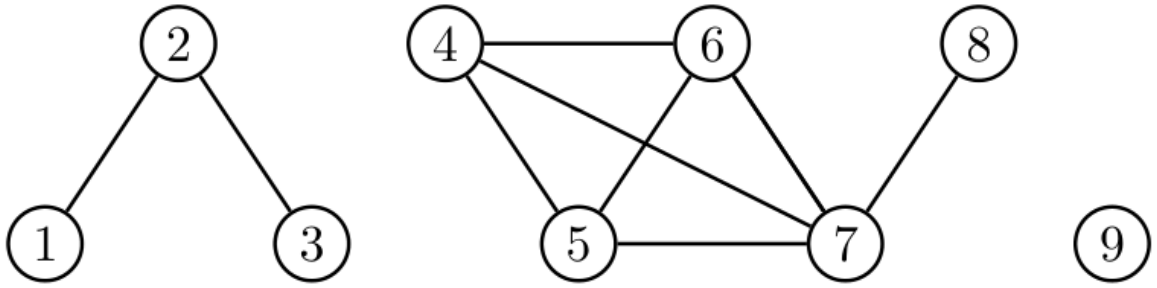
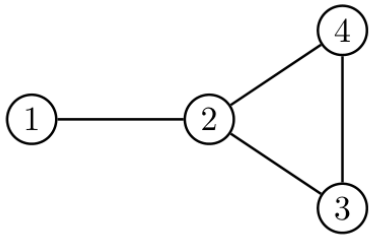


Figure 1: A Network with components $\{1,2,3\}, \{4,5,6,7,8\}, \{9\}$ *and maxclique cover* $\{1,2\}, \{2,3\}, \{4,5,6,7\}, \{7,8\}, \{9\}$.

A network is called an *interval network* if for any $ij \in G$ with $i < j$, we have $ik \in G$ and $kj \in G$ for all $i < k < j$. In an interval network, if an agent is linked with another agent, then he has to be linked any agent in between. The network in figure 1 is an interval network. In an interval network, we can assign to each agent i an interval I_i on real line such that $I_i \cap I_j = \emptyset$ if and only if $ij \in G$.

Bonacich-Katz Centrality Measure

Let G^k be the kth power of G[14]. The matrix G keeps track of direct links while G^k keeps track of indirect connections at length k and $g_{ij}^{[k]}$ show number of paths of length k between i and j. In the network represented in figure 2, for example, the number of paths of length 3 between agents 1 and 2 is $g_{12}^{[3]} = 3$ where the paths are {1, 2, 1, 2}, {1, 2, 3, 2}, {1, 2, 4, 2}. So, agent 1 can reach agent 2 in there different way by following exactly 3 links in network G. For any k, the ith coordinate of vector $G^k\mathbf{1}$ counts total number of paths of length



$$\mathbf{G} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad \mathbf{G}^2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

$$\mathbf{G}^3 = \begin{pmatrix} 0 & 3 & 1 & 1 \\ 3 & 2 & 4 & 4 \\ 1 & 4 & 2 & 3 \\ 1 & 4 & 3 & 2 \end{pmatrix} \quad \mathbf{B}\left(\mathbf{G}, \tfrac{1}{3}\right) = \begin{pmatrix} 2.4 \\ 4.2 \\ 3.6 \\ 3.6 \end{pmatrix}$$

Figure 2: Bonacich-Katz Centrality Measure

k originated from i where $\mathbf{1}$ is vector of ones[15]. For example, there are 5 paths of length 3 originating from agent 1 and 13 paths of length 3 originating from agent 2.

For a real number $\lambda \geq 0$ and a network G, we define n dimensional vector for Bonacich-Katz Centrality measure

$$B(G,\lambda) = \sum_{k=0}^{\infty} \lambda^k G^k \mathbf{1} \qquad 1$$

For each agent, Bonacich-Katz centrality score $B_i(G,\lambda)$ counts the total number of paths in G starting from i for all possible paths' lengths where the paths of length k are weighted by the geometrically decreasing factor λ^k. If the integer λ is small enough, then infinite sum in equation 1 converges to a finite value and $B(G,\lambda) = (I - \lambda G)^{-1}\mathbf{1}$. The upper bound for the integer λ is given by inverse of the largest eigenvalue of G (Debreu and Herstein (1953)).

[14] $G^0 = \mathbf{1}$ is set to be equal to identity matrix.
[15] Notice when $k = 0$, then $G^k\mathbf{1}$ reduces to vector of ones $\mathbf{1}$ and when $k = 1$, $G^k\mathbf{1}$ is simply degree vector; $G\mathbf{1} = D(G)$.

The Bonacich-Katz centrality measure how central, important, influential or popular an agent based on the location of agent in network. For the network[16] in figure 2, the centrality measures for agents are B(G, 1/3) = (2.4, 4.2, 3.6, 3.6) for $\lambda = \frac{1}{3}$. The centrality measures of agents 3 and 4 are same since their positions in the network are symmetric, whereas agent 2 emerges as most central agent since he can reach all agents in one step. But there are examples of social networks where agents have external sources of popularity or importance that are independent of agent's position in network such as quarterback of high school football team might have more popularity regardless of number and composition of his friends or in communication networks, an agent can have an external source of information that no one else has[17]. The definition in equation 1 ignore such differences by assigning a base value of importance 1 to each agent, that is, the vector of ones in the definition above implicitly assumes that each agent has same base value of importance, hence an agent has a higher centrality measure only because of his position in network. In equation 1, the vector $\mathbf{1}$ is replaced with a vector $\boldsymbol{u}$ that represents the external source of importance, influence of agents[18]in order to reflect external sources of centrality. This leads to following modified version of Bonacich-Katz Centrality measure:

$$B_{\boldsymbol{u}}(G,\lambda) = \sum_{k=0}^{\infty} \lambda^k G^k \boldsymbol{u} = (I-\lambda)^{-1}\mathbf{u} \qquad 2$$

Finally, we define a weight matrix H with $h_{ij} \in (0,1)$ for all i,j which reflects the weight of influence of agent j on agents i centrality. Replacing G with H in equation 2, gives us generalized Bonacich-Katz centrality measure

$$B_{\boldsymbol{u}}(H,\lambda) = \sum_{k=0}^{\infty} \lambda^k H^k \boldsymbol{u} = (I-\lambda H)^{-1}\boldsymbol{u} \qquad 3$$

III. SOCIALIZATION: NETWORK GAME

Each agent i is characterized by his type $\theta_i \in [0,1]$ and $\theta = (\theta_1, \dots, \theta_n)$ is the type profile. Therefore, we use "agent i" and "agent with type θ_i" as synonyms throughout the paper. Type can be interpreted as proclivity or tendency towards a certain behavior such as frequency of marijuana use, usage of alcoholic beverages, or attitudes such as educational aspiration or political position. Types are normalized to lie between 0 and 1.

[16] The inverse of largest eigenvalue for the network is 0.4608
[17] See Bonacich and Lloyd (2001)
[18] In our setting, since agents has ex-ante heterogeneities, we replace base vector $\boldsymbol{u}$ with type vector θ in the next section.

Each agent takes an action $x_i \in [0,1]$ which is interpreted as the actual behavior agent takes in the society. Let $x = (x_1, \dots, x_n)$ be the action profile. Given a network G and action profile $x = (x_1, \dots, x_n)$ utility of agent i is

$$u_i(x, G) = (1-\mu)\left(1 - \frac{1}{\tau_i}(x_i - \theta_i)^2\right) + \mu \sum\nolimits_j g_{ij}\left(1 - \frac{1}{\tau_i}\left(x_i - x_j\right)^2\right) \quad 4$$

An agent's utility consists of two parts: Coordination of his action with his own type and coordination of action with friends' actions. Coordination benefits are both normalized to 1. $\mu \in [0,1]$ is a parameter that determines the weight of coordination with friend against coordination with his own. As difference between his action x_i and his type θ_i and difference between his action x_i and friends' actions x_j's increases, agent utility decreases and τ_i determines the magnitude of decrease in utility. An agent's utility is weighted sum of coordination benefit with his own and sum of coordination benefit with his friends.

We define the network game as $\Gamma(G) = \{N, \{x_i\}_{i\in N}, \{u_i\}_{i\in N}\}$ where N is the set of players, $x_i \in [0,1]$ is the strategies of player i with the utility $u_i(x, G)$ where $x = (x_1, \dots, x_n)$ is the strategy profile.

Socialization focuses on influence of friends on one's behavior and leads similarities between friends. Hence agents under peer effect choose their actions based on their friends' actions which creates a strategic interaction between agents[19]. Therefore we, now, consider Nash equilibrium $x^\star(G)$ of network game which captures the strategic behavior of agents. But before turning our attention to details of Nash equilibrium, it is worth to consider two extreme cases. The first case is when $\mu = 1$. The utility of all agents in a network G reduces to $u_i(x, G) = \sum_j g_{ij}\left(1 - {}^1/_{\tau_i}(x_i - x_j)^2\right)$and there is continuum of Nash equilibria with $x_i^\star = x^\star$ for every $x^\star \in [0,1]$. The second case is when $\mu = 0$. Then utility of agents reduces $u_i(x, G) = 1 - {}^1/_{\tau_i}(x_i - \theta_i)^2$ and the Nash equilibrium strategy for all agents is to choose their type $x_i^\star = \theta_i$. When an agent i has no friends, that is, his utility is same with the latter case above and his optimal action is to choose his own type. Hence an agent's action is different then his type if he has some friend to coordinate and weight μ on coordination with friends is positive.

[19] Kandel (1978) finds that socialization is strongest in case of marijuana use and suggest that drug use is a particularly strategic behavior. See page 436.

For $\mu \in (0,1)$, there is unique Nash equilibrium and it is proportional to his generalized Bonacich-Katz Centrality measure which is expressed in the following theorem.

Theorem 1 *Network game $\Gamma(g)$ has a Nash equilibrium for any G. Moreoverif $\mu < 1$, then there is a unique Nash equilibrium*

$$x^{\star} = (1-\mu)B_u(h,\mu)$$

which is equal to generalized Bonacich-Katz centrality measure,

$$B_u(h,u) = \sum_{k=0}^{\infty} \mu^k \mathrm{h}^k \boldsymbol{u} = (\mathrm{I} - \mu\mathrm{h})^{-1}\mathbf{u}$$

where weight matrix h is defined as $h_{ij} = \frac{1}{1-\mu+\mu d_i(G)} g_{ij}$ and $u_i = \frac{1}{1-\mu+\mu d_i(G)}\theta_i$.

First part of theorem 1 follows from standard argument of existence of Nash equilibrium by verifying conditions of Kakutani's Fixed point theorem. Second part of theorem follows from Theorem 1 of (Ballester et al., 2006).

We also provide an alternative expression for Nash equilibrium actions that is different from the generalized Bonacich-Katz centrality measure, which keeps track of feedback effects. The Nash Equilibrium can be written as follows:

$$x^{\star}(g) = \boldsymbol{M}^{-1}\theta$$

where $\boldsymbol{M} = \boldsymbol{I} - \frac{\mu}{1-\mu}(g - \boldsymbol{D})$ is an M-Matrix[20] and $\boldsymbol{D}$ is the diagonal degree matrix.

Let $\boldsymbol{N} \coloneqq \boldsymbol{M}^{-1}$ and notice that each row of both $\boldsymbol{M}$ and $\boldsymbol{N}$ adds up to 1. Then, in the alternative expression of Nash equilibrium, the equilibrium action of any agent i is a weighted average of types of all agents (including himself) and n_{ij} represent the weight of agent j's type θ_j on agent i's equilibrium action $x_i^{\star}(g)$. Moreover, an agent i assigns highest weight on his type independent from the network g[21], that is, $n_{ii} \geq n_{ik}$ for all $k \in N$ for all g.

The best response of each agent in network game is to choose an action which is weighted average of one's type and average action of his friends in network:

$$x_i^{\star}(x_{-i}) = \frac{1-\mu}{1-\mu+\mu d_i(G)}\theta_i + \frac{\mu}{1-\mu+\mu d_i(G)}\sum_j g_{ij}x_j \qquad 5$$

[20]See appendix for details.
[21] See Theorem 3.2(i) of McDonald et al. (1995) and note that $\boldsymbol{M}$ is symmetric so does $\boldsymbol{N}$.

which reflects friend's effect on an agent's action through socialization process. An agent's optimal action is to choose a weighted average of his own type and sum of action of his friends. Agent i assigns weight $1-\mu$ on his type and assigns weight μ on each of his friends and total weights assigned adds up to $1-\mu+\mu d_i(G)$. As μ, the weight of coordination benefits with friends in the utility function, increases, agent's action gets closer to average actions if his friends.

Moreover, in any network G, sum of equilibrium actions equals to sum of types.

Proposition 2 *Let* $\mu < 1$ *and* $x^\star(G)$ *be the Nash equilibrium of network game* $\Gamma(G)$*, then* $\sum_{i\in N} x^\star(G) = \sum_{i\in N} \theta_i$.

An immediate corollary of the proposition above is difference between equilibrium actions and average type, $\bar{\theta} = \frac{1}{n}\sum_{i\in N} \theta_i$, adds up to 0. For all agents whose equilibrium action is higher than average type, there are agents whose equilibrium actions are lower than average type. That is to say that average equilibrium action equals to average type in society in all networks.

Corollary 3 *Let* $\mu < 1$ *and* $x^\star(G)$ *be the Nash equilibrium of network game* $\Gamma(G)$*, then* $\sum_{i\in N}[x_i^\star(G) - \theta_i] = 0$.

IV. SELECTION: NETWORK FORMATION

In previous section, we consider an agent's optimal action for a given network and show the effects of friends on an agent's decision. Because of socialization, agents choose actions that are different than their types which would be their optimal actions if they have no friends or there is no socialization effects. In fact, agent choose their action to increase similarities between their actions and their friends' actions which capture the idea of socialization. But they are not allowed to decide who their friends should be if they are to interact with them to coordinate on certain behavior. In this section, we consider selection aspect of social networks in which agents decide to break a relationship because there is a big enough difference in their actions and they would be better off from ending relationship or agents form new relationships if they have high enough similarities. Agents make decisions to add or delete a link based on the utility of network which is set to be equal to the utility of that agent in the network game under Nash equilibrium actions:

$$U_i(G) = u_i(x^\star(G), G) \qquad 6$$

The utility of network incorporates the socialization effects, that is, when an agent adds or deletes a link, the utility of agents changes not just because the network structure is now different but the equilibrium actions are also changed. The utility $U: \mathcal{G} \to \mathrm{R}^{\mathrm{n}}$ allows agents to rank all possible networks so that they can decide if starting or ending a friendship is beneficial for them.

In friendship networks, forming a new friendship requires consent of both agent in the sense that both agents should benefit from new friendship, but agents can unilaterally end friendship if they alone benefit from it. Pairwise stability capture this characteristic of friendship networks. A network G is said to be pairwise stable if

(i) for all $ij \in G$, $U_i(G) \geq U_i(G - ij)$ and $U_j(G) \geq U_j(G - ij)$, and

(ii) for all $ij \in G$, if $U_i(G + ij) > U_i(G)$, then $U_j(G + ij) < U_j(G)$.

First condition in the definition requires that deleting a link in a pairwise stable network cannot (strictly) increase utility of any agent involved in the link[22]. Second condition concerns adding links. It requires that if utility of an agent strictly increases by adding a new link in pairwise stable network, then the other agent involved in the link must be (strictly) worse off[23]. In other words, in a pairwise stable network, no single agents can increase his utility by deleting a link and no two agents can mutually increase their utilities by adding a link.

Existence of Pairwise Stable Networks

Known results in literature about the existence of pairwise stable networks do not come in our aid since they do not apply in our model. Earliest result about the existence of pairwise stable networks is by Jackson and Watts (2001) where they showed that there is (at least one) pairwise stable network or a closed cycle[24] for any utility function $U: \mathcal{G} \to \mathrm{R}^{\mathrm{n}}$.

[22] Notice that if a link between two agents does not change their utilities if it is added or deleted, then the definition requires presence of this link in pairwise stable network. In this sense, the definition above favors bigger networks when a link has no effect on agents involved in this link.

[23] If a network satisfies only the condition (i) in the definition above, then it is called *link-deletion proof* network and if it satisfies only the condition (ii), then it is called *link-addition proof* networks. A pairwise stable network is both link deletion proof and link addition proof.

[24] A closed cycle is a subset of networks $C \subset \mathcal{G}$ such that for any network $G \in C$, there does not exist an improving path that leads to another network $G' \in C$ that does not belong the closed cycle where an improving path is a sequence of networks on which either a link is broken if it increases utility of any agent involved in the link or a link is added if it increases utilities of both agents.

They also showed that there are no cycles if there exists a potential function $\omega: \mathcal{G} \to \mathrm{R}$ such that $\omega(G') > \omega(G)$ if and only if G' is an improvement on G and G', G differs only in one link. They provided examples of potential function for some earlier papers and suggest that a natural candidate for potential function is the sum of utilities: $\omega(G) = \sum_i U_i(G)$. But in our model, it is difficult to keep track of equilibrium actions as there are changes in the network structure because of feedback effects on equilibrium actions. A change in network leads changes in equilibrium actions (hence utility) of any agent that belongs to same component. Moreover, even if agents i, j prefers $G + ij$ to G, it can lead a decrease in the utility of other agents so that sum of utilities may decrease even if $G + ij$ is an improvement on G.

Jackson and Watts (2001) does not provide any assumption on the primitives of model that guarantees existence of pairwise stable networks. Hellmann (2009), however, demonstrate the existence of pairwise stable networks by assuming certain conditions on utility function of networks. In particular, he shows that if utility function satisfies convexity in own links and strategic complementarity, then there does not exist a closed cycle and there exists a pairwise stable network. Convexity in own links implies that an agent's marginal utility of any link (ij) does not decrease if agent agent has more friends and strategic complementarity implies that an agent's marginal utility of any link does not decrease if other agents add links that does not involve the agent.

The result of Hellmann (2009) does not apply does not apply to our model. The utility function in our model does not satisfy strategic complementarity. To see this, consider three contiguous agents$i, i + 1, i + 2$ in an empty network. Marginal utility of agent i of a link with i+1 decreases if $i + 1$ and $i + 2$ add a link between each other which shows utility function does not satisfy strategic complementarity. The reason for the decrease in marginal utility of i when $i + 1$ and $i + 2$ are friends is $i + 1$ has to choose an action that is higher than his type which reduces the coordination benefits of i with agent $i + 2$.

Agents, in our model, are characterized by their types. Marginal utility of a certain links essentially depends on the coordination benefits of two agents involved in this link which in turn depends on the optimal actions of agents, not on the number of friends agents have. If other agents add links in a way that the actions of two players are now closer, then marginal utility of certain link can only increase as strategic complementarity requires.

Partial characterization of pairwise stable networks

Although, we cannot show existence of pairwise stable networks, we have some partial characterization of pairwise stable networks when parameters of model μ and τ satisfy certain conditions. Notice that, the magnitude of change in utility when a link is added or deleted depends on not only how optimal actions change but also value of tolerance rate τ_i. Nash equilibrium actions are independent of tolerance rates which, however, play a critical role in selection process. To simplify the analysis, we assume that tolerance rates are same for all agents and equidistance type profile:

- $\tau_i = \tau$ for all i, and,
- $\theta_1 = 0$ and $\theta_{k-1} - \theta_k = \frac{1}{n-1}$ for $k = 1, \dots, n-1$.

We say agents k and $k+1$ are contiguous agents.

We start by considering two extreme cases of networks: empty network where no two agents are linked and complete network where there is a link between any two agents. The following proposition shows that empty network is pairwise stable when tolerance rate is small enough and complete network is pairwise stable when tolerance rate is high enough.

Proposition 4 Suppose $\mu \in (0,1)$. Empty network is pairwise stable if and only if $\tau < \frac{1-\mu}{(1+\mu)^2}\frac{1}{(n-1)^2}$.

In an empty network, contiguous agents have highest incentive to form a link. When they form a link, the equilibrium actions will be closer than the difference in their types: $\frac{1}{n-1}$. Whether they have incentive to add the link depends on the magnitude of the loss in coordination utility with their own selves since equilibrium action is, now, different than their type and the magnitude of gain in coordination benefit with friend. If tolerance rate is small enough, the loss in coordination benefit with one's self is higher that the gains in coordination benefit with others. Threshold on tolerance rate a decreasing function of μ. As μ increases, there are two affects: the equilibrium actions of contiguous agents gets closer to each other and the weight on the loss of coordination with oneself decreases. As μ converges to 1, then the threshold converges to 0.

In the next proposition, we show the intervals for tolerance rates so that certain subset of networks where the degree is at most 2 are pairwise stable. First set of networks, $\mathcal{I}_1$, is the subset of interval networks such that for all $G \in \mathcal{I}_1$, we have $d_i(G) \leq 1$ for all i and if

$d_k(G) = 0$ for some k, then both $d_{k-1}(G) = d_{k+1}(G) = 1$. Notice that if $d_i(G) = 1$ for some i, then we have either $g_{i-1,i} = 1$ or $g_{i,i+1} = 1$. We also call this class of networks as dyadic networks since all components are either singleton or contains two dyads that are contigous agents.

Second subset of network, $\mathcal{I}_2$, is a subset of interval networks such that for all $G \in \mathcal{I}_2$, we have $d_i(G) \leq 2$ and

- if $d_k(G) = 2$ for some k, then both $g_{k-1,k} = 1$ and $g_{k,k+1} = 1$,
- if $d_k(G) = 0$ for some k, then both $d_{k-1}(G) = 1$ and $d_{k+1}(G) = 1$.

The following proposition shows the conditions on tolerance rate and weight on coordination, so that, the subsets of networks described above are pairwise stable.

Proposition 5 Suppose $\mu \in (0,1)$.

1. If $\frac{1-\mu}{(1+\mu)^2}\frac{1}{(n-1)^2} \leq \tau < (1-\mu)\frac{1}{(n-1)^2}$, then any $g \in \mathcal{I}_1$ is pairwise stable.
2. If $< (1-\mu)\frac{1}{(n-1)^2} \leq \tau < \frac{4(1-\mu)(1-\mu^2)}{(1+2\mu)^2}\frac{1}{(n-1)^2}$, then any $g \in \mathcal{I}_2$ is pairwise stable[25].

Notice the lower threshold in proposition 5.1 is the threshold for empty network in proposition 4. So, when tolerance rates exceed the upper bound of empty network, two contiguous agents have incentive to form a link. Upper threshold on tolerance rate on part 1 prevent an agent to form a link with the other contiguous agent. Therefore, in any network in $\mathcal{I}_1$, the components are either consist of two contigous agents or one agent and there can not be two contigous agents with no friends.

When the tolerance rates increase further, an agent who has already has a friend has, now, incentive to add a link with the other contiguous agent, that is, there is a component with 3 contiguous agents where the middle agent act as central agent connected to two other agents.

In the results in proposition 4 and proposition 5, the tolerance rate plays an important role. When tolerance rate is small enough, the agents do not have incentive to add any link so that empty network is the only pairwise stable network. As tolerance rates increases, agents start friendship with agents who are closest in types, i.e., contiguous agents. As tolerance rate

[25] Note that $(1-\mu)\frac{1}{(n-1)^2} \leq \frac{4(1-\mu)(1-\mu^2)}{(1+2\mu)^2}\frac{1}{(n-1)^2}$ if and only if $\mu \leq \frac{\sqrt{7}-1}{4}$.

increases, any dyadic network $G \in \mathcal{I}_1$become pairwise stable. In dyadic networks, if an agent has a friend, his friend is a contiguous agent, that is, his friend is one of the agents who is closest in terms of types. In other words, similarity in terms of types leads friendship as selection states. Moreover, we cannot have two contiguous agents without friend, that is, if an agent has no friends in a dyadic network, it means the contiguous agents have already friends and they arenot willing to be friend with the agents if tolerance rate is smaller than $(1-\mu)\frac{1}{(n-1)^2}$. However, if tolerance rate is higher than this level, they are willing to form the link and the network belongs to $\mathcal{I}_1$.

In addition, all the bound in results above, are decreasing function of μ and as μ converges to 1, the all bounds converge to 0. In all the networks considered so far, as μ increases the differences between actions of friends gets smaller and converges to 0. Therefore, as weight on coordination increases, tolerance rate does not need to be as high for a network to be pairwise stable. The effect of an increase in μ on utility is twofold. First, as equilibrium actions gets closer to each other, the coordination benefits get higher and the weight on coordination benefits gets higher although the loss in coordination with own is now higher, agent assigns a lower weight on coordination with own type.

Pairwise stable network when n=5

In this section, we examine parameter values for which there are pairwise stable networks exist and structure of those networks. Although partial characterization we provided in the previous section gives the general idea of how pairwise stable networks change as the parameters change. But it will be much clearer to see these changes at a fixed value such as n = 5.

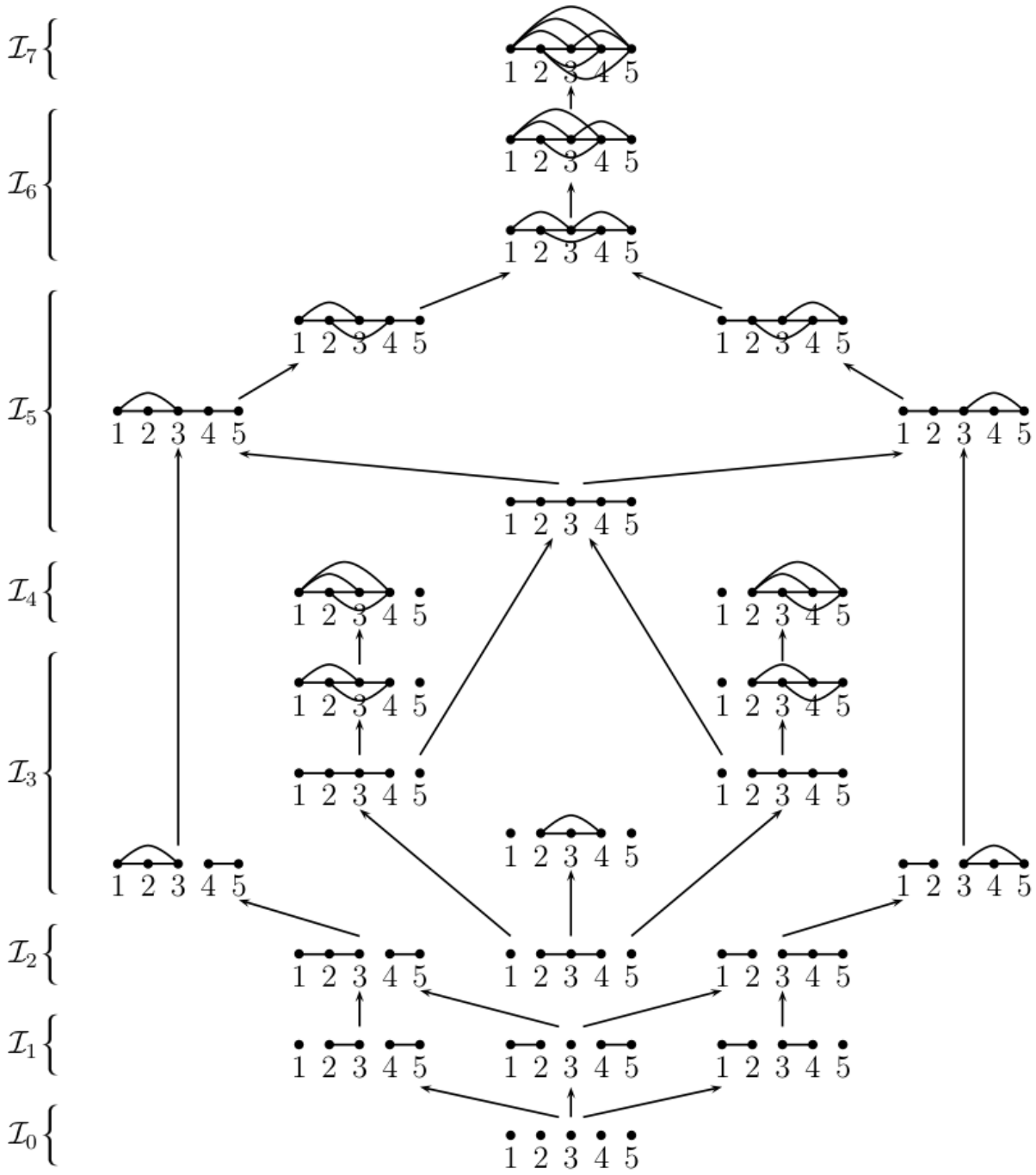


Figure 3 Pairwise stable networks for n=5.

Figure 3 shows classes of pairwise stable networks where agents are shown on a line. Agent 1 with $\theta_1 = 0$ are shown on the left of each line and agent 5 with $\theta_5 = 0$ are shown on right of each line. The lines between agents represent existence of links between those agents. For example, in an empty network, $\mathcal{I}_0$, there are no lines between any agents whereas in dyadic networks, $\mathcal{I}_1$, there are links between contiguous agents.

The arrows between pairwise stable networks show the evolution of pairwise stable network starting form empty network to complete network as the parameters of the model increases. There are 8 classes of interval networks, $\mathcal{I}_1$ and $\mathcal{I}_2$ corresponding the class of network in previous section. Notice that, expect between classes $\mathcal{I}_0$ and $\mathcal{I}_1$, there are only one links are one links are added as one moves up following the arrows. In general, the direction of arrows between networks shows whether there are incentives for agents involved in the links between networks to form a links, but in Figure 4, the arrows show the evolution of networks are parameters of model increases.

Figure 5 shows parameter values for which classes of interval networks are pairwise stable. It shows values of tolerance rates, τ, on vertical axis and peer influence, μ, on horizontal axis. For each pairwise stable interval network classes, values for which those classes are pairwise stable are shown with green whereas all the values of all previous classes are shown with blue.

For lowest values of parameters of model, empty network is pairwise stable. Later dyadic networks and networks in class $\mathcal{I}_2$ become pairwise stable as shown in previous section. As parameters increase, there are two paths pairwise stable networks take. In first, agents add links within their components eventually turning those components complete subnetworks and forming a clique. In the second path, agents form a link with someone in another component, expanding the sizes of components. These paths can be best seen in class $\mathcal{I}_2$. In the two pairwise stable networks on the left and right, all agents belong to two components and those in same components, they coordinate their actions so that they have the highest incentive to form a link within their components. Therefore, as parameters increase, they form links within their components and forming a clique. For the middle pairwise stable network in class $\mathcal{I}_2$, however, agents 1 and 5 with extreme types are left out of the component. As parameters increase, either individuals in not-trivial component from a link within their component or either agents 1 or 5 joins the component and increases the size of the component. Those different paths pairwise stable networks can take can be seen at the middle section of class $\mathcal{I}_3$ and $\mathcal{I}_4$.

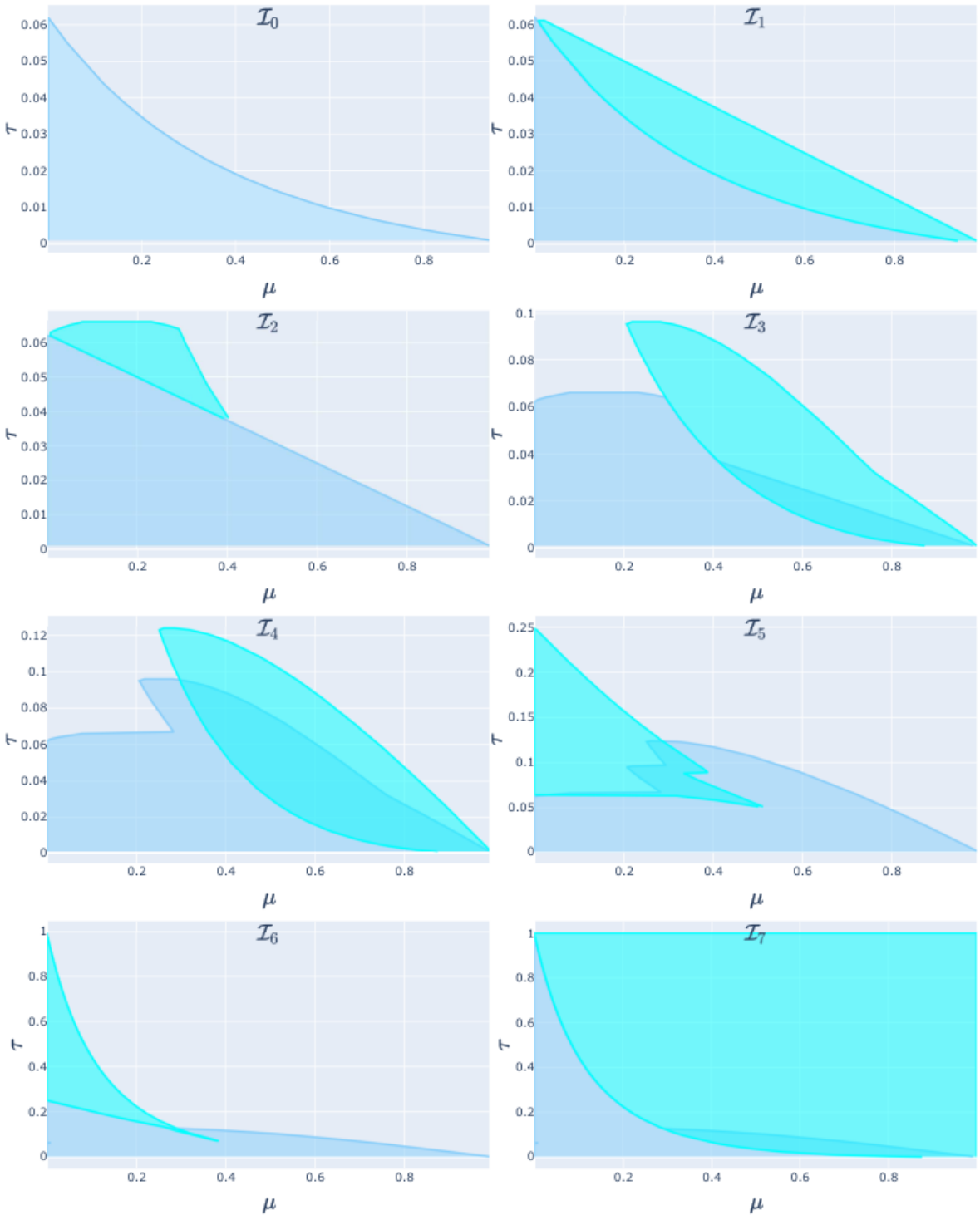


Figure 6 Parameter Values for classes of pairwise stable network

In class $\mathcal{I}_5$, all agents belong to a single component. In the bottom of class $\mathcal{I}_5$, only contiguous agents are linked which is the configuration with least number of links to form a single component. Form this point on, agents form links starting with closest in type and eventually complete network is pairwise stable.

In our model where we examine the effects of selection and socialization on structures of social networks, as can be seen in Figure 4 where pairwise stable networks are shown, agents make friends with those whose types are closest to them. Any social network in which an agent skips the individual whose type is closest to them and makes friends with others ispairwise stable. This situation arises from the selection which claims that the similarities between individuals who are friends with each other are formed by the individuals' own preferences. In addition, when agents make friends with contiguous agents, as predicted by the socialization, their equilibrium behaviors will differ from their own types and converge to each other. However, either with the increase in the tolerance rate or increase in the strength of the peer effect, the agents will make friends with the agents closest to them in types. After this stage, two types of effects will begin to show themselves. On the one hand, the connections between the components will increase and these components will eventually form a clique by establishing a full network among themselves or the components will grow by establishing connections with the individuals closest to these components. In the first case, as the connections between the components increase and they move towards forming a clique, the behaviors will converge and the observed similarities will increase due to the socialization effect. Of course, in cases where the society is divided into more than one component, the difference between the average behaviors of these components will increase. However, as we showed in Proposition 2, these differences will not have an effect on the average behaviors of the entire society. In other words, the average behaviors of the society will always be equal to the average type that shows the given characteristics of the society. In the second case, the components will grow by connecting with the individuals closest to them. Eventually, the entire society will be in a single component, and with increasing tolerance effects or increasing peer effects, the pairwise stable network will be the complete network.

V. CONCLUSION

We consider a friendship network where agents where agents have two decisions to make: they decide on an action which is of mutual importance with their friends and they decide on their friends. Agents decide on their actions to minimize the difference between their action and friends' actions while trying to minimize the difference between their action and their type. The types of agents reflect their tendencies towards the action and are the optimal actions of agents when they have no friend. An agent's action, then, depends on their type and their friends' actions which creates an environment of strategic interaction. We consider a network game that captures this strategic interaction and showed that there is

always a Nash equilibrium and it can be expressed a function of generalized Bonacich-Katz centrality measure which essentially keeps track the feedback effects on an agent's action that populates through network. We also showed that the average equilibrium action equals to average types in any network and for any values of tolerance rate and weight on coordination.

We, then, consider friendship decision of agents. In friendship network, agents require mutual consent in order to start a relationship while they can unilaterally end a relationship. This aspect of friendship networks is captured by definition pairwise stability which is the solution concept we used. The utility an agent gets in a network is set to be the utility of agent in network game under Nash equilibrium. The structure of pairwise stable networks depends on the value of tolerance rate. When tolerance rate is low enough, then empty network is pairwise stable where no agent has incentive to form a link. On the contrary, when tolerance rate is high enough, complete network is pairwise stable. In between, we show two class of networks as pairwise stable depending on tolerance rate. We, then, show all pairwise stable networks when n=5. When the parameters of the model, tolerance rate and peer effect, increase, starting from empty network, agents start to form link with those who are closest in type as selection process predicts. Due to socialization, agents coordinate their behaviors and have higher incentive to form relationships within their component eventually forming a clique. For large enough parameters, however, agents will have incentive to form relationships with adjacent components and after this process continues, the complete network will be pairwise stable.

Appendix

Before moving to proof of Theorem 1, we will discuss the condition $\mu < 1$ in the second part of the theorem. Given a network $g$and corresponding weight matrix h, let us define spectral radius $\rho(h) = max\{|\lambda|: \lambda\ is\ an\ eigen\ value\ of\ h\}$ of matrix h. The infinite sum of geometric series $\sum_{k=0}^{\infty} \mu^k h^k$ converges if $\mu < \frac{1}{\rho(h)}$ and it is equal to $(I - \mu h)^{-1}$ if it converges. Theorem $III^{\star}$ of Debreu and Herstein (1953) establishes that $(I - \mu h)^{-1}$ is well-defined and non-negative if $\mu < \frac{1}{\rho(h)}$. To be able to express Nash equilibrium in terms of Bonacich-Katz centrality measure, we need the condition $\mu < \frac{1}{\rho(h)}$.

Notice that $\rho(H)$ depends on both g and μ. It turns out that $\mu < \frac{1}{\rho(h)}$ if and only if $\mu < 1$.

Lemma 6 *Let G be any network different from the empty network. Let h be the corresponding weight matrix for some μ. Then $\mu < \frac{1}{\rho(h)}$* if and only if $\mu < 1$.

Proof. Let $G \in \mathcal{G}$ be a network different than empty network and$\mu < 1$. Let $d_{min} = \min_i d_i(G)$ and $d_{max} = \max_i d_i(G)$. Notice that $d_{min}, d_{max} \geq 1$ since G is different than empty network.

The corresponding weight matrix h is a non-negative matrix. By Theorem 8.1.22 of Horn and Johnson (1985)[26], we have

$$\frac{d_{\min}}{1-\mu+\mu d_{\min}} \leq \rho(h) \leq \frac{d_{\max}}{1-\mu+\mu d_{\max}}$$

Note that the ratio $\frac{d}{1-\mu+\mu d}$ is increasing in d (so that upper-bound is higher than lower-bound) and if there exist an agent j with $d_j(g) = 0$ (so $d_{\min} = 0$), then lower bound of spectral radius does not bind. Then, we have

$$\frac{1-\mu+\mu d_{\max}}{d_{\max}} \leq \frac{1}{\rho(h)}$$

But we have $\mu < \frac{1-\mu+\mu d_{\max}}{d_{\max}}$ if and only if $\mu < 1$ which establishes $\mu < \frac{1}{\rho(h)}$if and only if $\mu < 1$.

Proof of Theorem 1. To see existence of Nash equilibrium, observe that strategy set $[0,1]$ is convex, compact for each agent. The utility function u_i is continuous in x. u_i is strictly concave in x_i unless $\mu = 1$ and $g_{ij} = 0$ for all j. By referring Maximum Theorem[27], we establish that best response correspondence is single valued upper semi-continuous correspondence (or simply a continuous function). The existence of Nash equilibrium follows immediately.

[26] For your reference, we include the statement of theorem here.
Let $A \in R^{n\times n}$ and A is non-negative matrix. Then

$$\min_{1\leq i\leq n} \sum_{j=1}^{n} a_{ij} \leq \rho(A) \leq \max_{1\leq i\leq n} \sum_{j=1}^{n} a_{ij}$$

and

$$\min_{1\leq j\leq n} \sum_{j=1}^{n} a_{ij} \leq \rho(A) \leq \max_{1\leq j\leq n} \sum_{j=1}^{n} a_{ij}$$

[27] An interested reader can refer Theorem 9.17 in Sundaram (2005).

If $\mu \neq 1$, then Nash equilibrium is unique since u_i is strictly concave in x_i. If $\mu = 1$, then there is a continuum of equilibriums $x^\star = x_i^\star \in [0,1]$ and if for some i, we have $g_{ij} = 0$ for all j, then it is obvious that $x_i^\star = \theta_i$.

Fix the network g and let $\mu < 1$. We have existence and uniqueness of Nash equilibrium, $x^\star$, from argument above.

If $x^\star$ is in interior of strategy space, then it satisfies $\frac{\partial u_i}{\partial x_i}(x^\star) = 0$ for all i, that is,

$$\frac{\partial u_i}{\partial x_i}(x^\star) = \frac{-2}{\tau_i}\left[(1-\mu)(x_i^\star - \theta_i) + \mu \sum_j g_{ij}(x_i^\star - x_j^\star)\right] = 0$$

Rewriting the equation above gives us,

$$[1 - \mu + \mu d_i(g)]x_i^\star = (1-\mu)\theta_i + \mu \sum_j g_{ij}\, x_j^\star$$

or,

$$x_i^\star = \frac{1-\mu}{1-\mu+\mu d_i(g)}\theta_i + \frac{\mu}{1-\mu+\mu d_i(g)}\sum_j g_{ij} x_j^\star$$

Let $h_{ij} = \frac{\mu}{1-\mu+\mu d_i(g)}\, g_{ij}$ be the weight matrix and $u_i = \frac{1-\mu}{1-\mu+\mu d_i(g)}\theta_i$. Then the equation above can be rewritten in matrix form as

$$x^\star = (1-\mu)\boldsymbol{u} + \mu \boldsymbol{h}\boldsymbol{x}$$

$$[\mathbf{I} - \mu \boldsymbol{h}]x^\star = (1-\mu)\boldsymbol{u}$$

$$x^\star = (1-\mu)[\mathbf{I} - \mu \boldsymbol{h}]^{-1}\boldsymbol{u}$$

where last line follows from Lemma 8 and Theorem $III^\star$ of Debreu and Herstein (1953). Hence, we can rewrite Nash Equilibrium as $x^\star = (1-\mu)B_\theta(h,\mu)$.

To see that $x_i^\star \in [0,1]$ for all i, let $\underline{\theta} = \min\{\theta_1, \dots, \theta_n\}$ and $\overline{\theta} = \max\{\theta_1, \dots, \theta_n\}$. Notice that $\mu\underline{\theta}\mathbf{1} < \mu^k \boldsymbol{h}^k \theta < \mu\overline{\theta}\mathbf{1}$ for all $k > 0$[28]. Then the vector of Bonacich-Katz measures is bounded

[28]Strict inequalities turn to weak inequality when k = 0.

$$\frac{1}{1-\mu}\underline{\theta}\mathbf{1} = \sum_{k=0}^{\infty}\mu^k\underline{\theta}\mathbf{1} < \sum_{k=0}^{\infty}\mu^k\boldsymbol{h}^k\boldsymbol{\theta} < \sum_{k=0}^{\infty}\mu^k\overline{\theta}\mathbf{1} = \frac{1}{1-\mu}\overline{\theta}\mathbf{1}$$

which establishes $x_i^\star \in [0,1]$ for all i since $x^\star = \sum_{k=0}^{\infty}\mu^k\boldsymbol{h}^k\boldsymbol{\theta}$. This verifies that $x^\star$ is in interior of strategy space.

Now we provide an alternative representation of equilibrium actions different than Bonacich-Katz in Theorem 1. But first, we need to introduce M-matrix.

Definition 7 *A square matrix* $\boldsymbol{A} \in R^{n\times n}$ *is called M-Matrix if* $\boldsymbol{A} = s\boldsymbol{I} - \boldsymbol{P}$ *for some* $\boldsymbol{P} \geq 0$ *and* $s > \rho(\boldsymbol{P})$. *If, in addition,* $\boldsymbol{A}$ *is a symmetric, then it is called Stieltjes matrix.*

Fix a network g and assume $\mu < 1$. Notice that the best-response function

$$[1-\mu+\mu d_i(g)]x_i^\star = (1-\mu)\theta_i + \mu\sum_j g_{ij}\,x_j^\star$$

can be written in matrix form as

$$\left[\boldsymbol{I} - \frac{\mu}{1-\mu}(g-\boldsymbol{D})\right]x^\star(g) = \theta$$

where $\boldsymbol{D} = \boldsymbol{I} \star d(g)$ is the diagonal degree matrix. Let $\boldsymbol{M} = \boldsymbol{I} - \frac{\mu}{1-\mu}(g-\boldsymbol{D})$ and observe the followings

$M_{ii} = 1 + \frac{\mu}{1-\mu}d_i(g)$ for any $i \in N$,

$M_{ij} = -\frac{\mu}{1-\mu}g_{ij}$ for any $i \neq j$

and if $\boldsymbol{M}^{-1}$ exist and positive, then we have the existences of Nash equilibrium strategies and Nash equilibrium is in the interior of $[0,1]$. The following Theorem shows that if $\boldsymbol{M}$ is a M-Matrix, then its inverse exists and positive. See Plemmons (1977) for a survey of properties of M-matrix in the of spectral Theorem below.

Theorem 8 *Let* $\boldsymbol{M}$ *be a M-Matrix. Then* $\boldsymbol{M}$ *is inverse-positive, that is,* $\boldsymbol{N} := \boldsymbol{M}^{-1}$ *exist and* $\boldsymbol{M}^{-1} \geq 0$.

To show **M** constructed above is a M-Matrix, we need to show that it can written as $\mathbf{M} = \mathrm{s}\mathbf{I} - \mathbf{P}$ for some $\mathbf{P} \geq 0$ and $\mathrm{s} > \rho(\mathbf{P})$. Set $s = 1 + \frac{\mu}{1-\mu}n$ and construct $\boldsymbol{P} \geq 0$

$P_{ii} = \frac{\mu}{1-\mu}(n - d_i(g))$ for any $i \in N$ and

$P_{ij} = \frac{\mu}{1-\mu} g_{ij}$ for any $i \neq j$.

By Theorem 8.1.22 of Horn and Johnson (1985) in footnote 26 and by construction of $\boldsymbol{P}$, we have

$$\frac{\mu}{1-\mu} n \leq \min_{1 \leq i \leq n} \sum_{j=1}^{n} P_{ij} \leq \rho(\boldsymbol{P}) \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} P_{ij} = \frac{\mu}{1-\mu} n$$

That is, we have $1 + \frac{\mu}{1-\mu} n = s > \rho(\boldsymbol{P}) = \frac{\mu}{1-\mu} n$. Therefore $\boldsymbol{M}$ is a M-Matrix and equilibrium actions can be written as

$$x^{\star}(g) = \boldsymbol{M}^{-1}\theta = \left[\boldsymbol{I} - \frac{\mu}{1-\mu}(g - \boldsymbol{D})x^{\star}(g)\right]^{-1} \theta$$

Proof of Proposition 2. Let $\mu < 1$ and $G \in \mathcal{G}$ be any network. Let $x^{\star}$ be the Nash Equilibrium of $\Gamma(G)$. Then, for all i, we have

$$(1-\mu)x_i^{\star} + \mu d_i(G)x_i^{\star} = (1-\mu)\theta_i + \mu \sum_j g_{ij}\, x_j^{\star}$$

Rewriting the equation above gives

$$(1-\mu)(x_i^{\star} - \theta_i) = \mu \sum_j g_{ij}\,(x_j^{\star} - x_i^{\star})$$

and summing the equation above for all i, we have

$$(1-\mu)(x_i^{\star} - \theta_i) = \mu \sum_j g_{ij}\,(x_j^{\star} - x_i^{\star}) = 0$$

where last equality follows since the differences in equilibrium actions of any two friends $ij \in G$ add up to 0.

Proof of Proposition 4. Let $G^{\emptyset}$ be the empty network and let $G = G^{\emptyset} + ij$ be the network with only one link between i and j. WLOG assume that $i < j$.

Notice that $x^{\star}(G^{\emptyset}) = \theta_i$ and $U_i(G^{\emptyset}) = 1 - \mu$ for all i. For network G, we have $x_k^{\star}(G) = \theta_k$ for $k \neq i, j$ and $x_i^{\star}(G) = \frac{1}{1+\mu}(\theta_i + \mu\theta_j)$ and $x_j^{\star}(G) = \frac{1}{1+\mu}(\theta_j + \mu\theta_i)$ and the change in utility $\Delta U = U_i(G) - U_i(G^{\emptyset}) = U_j(G) - U_j(G^{\emptyset})$ is

$$\Delta U = (1-\mu)\left[-\frac{1}{\tau}\frac{\mu^2(\theta_i - \theta_j)^2}{(1+\mu)^2}\right] + \mu\left[1 - \frac{1}{\tau}\frac{(1-\mu)^2(\theta_i - \theta_j)^2}{(1+\mu)^2}\right]$$

$G^{\emptyset}$ is pairwise stable if and only if $\Delta U < 0$. A little bit algebra shows that

$$\Delta U < 0 \Leftrightarrow \tau < \frac{1-\mu}{(1+\mu)^2}(\theta_i - \theta_j)^2$$

The upper bound on τ is minimized when $j = i + 1$, then it is not optimal for any agent to even form a link with another agent whose type is closest. But, then, the condition reduces to $\tau < \frac{1-\mu}{(1+\mu)^2}\frac{1}{(n-1)^2}$ which proves the proposition.

Proof of Proposition 5. Let $\mu \in (0, 1)$.

1. Let g be a network satisfying the conditions in statement of proposition. We need to show that there are no incentives to break an existing link nor form a new link. Suppose $i, j \in g$, then $j = i - 1$ or $j = i + 1$. Since $\tau \geq \frac{1-\mu}{(1+\mu)^2}(\theta_i - \theta_j)^2$, breaking the link will reduce utility of both i and j.

   To show that there are no incentives to add a link, we need to consider two cases:

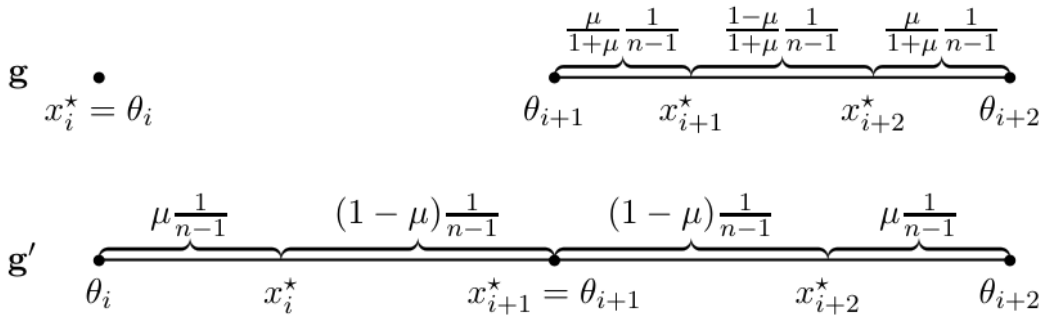
   - $d_i(g) = 0$, or
   - $d_i(g) = 1$.

   Consider the first case. Then both $d_{i-1}(g) = 1$ and $d_{i+1}(g) = 1$. Without loss of generality, assume $j = i + 1$ and consider the network $g' = g + ij$. Then we have

$x_i(g) = \theta_i$	$x_i(g') = \theta_i + \mu\frac{1}{n-1}$

$x_{i+1}(g) = \theta_{i+1} + \frac{\mu}{1-\mu}\frac{1}{n-1}$	$x_{i+1}(g') = \theta_{i+1}$
$x_{i+2}(g) = \theta_{i+2} - \frac{\mu}{1-\mu}\frac{1}{n-1}$	$x_{i+2}(g') = \theta_{i+2} - \mu\frac{1}{n-1}$

The result above shown the graph below where a line between agents represents a link. The change in utilities is



$$\Delta U_i = U_i(g') - U_i(g) = \mu - \frac{1}{\tau}\frac{\mu(1-\mu)}{(n-1)^2}$$

$$\Delta U_{i+1} = U_{i+1}(g') - U_{i+1}(g) = \frac{1}{\tau}\frac{\mu(1-\mu)}{(n-1)^2} - \left[\frac{1}{(1+\mu)^2} - (1-\mu)\right]$$

Then g is pairwise stable if $\Delta U_i < 0$ or $\Delta U_{i+1} < 0$. But $\Delta U_i < 0$ if and only if $\tau < \frac{1-\mu}{(n-1)^2}$ and $\Delta U_{i+1} < 0$ if and only if $\tau < \frac{\sqrt{5}-1}{2} \cong 0.618$ which gives the upper-bounds in proposition.

Now consider second case: $d_i(g) = 1$. Suppose $g_{i,i+1} = 1$. Let $d_{i-1}(g) = 1$ since other case is considered above. Calculations shows that this case holds for any $\tau \geq 0$.

2. Let $g \in \mathcal{J}_2$. We need to show that there are no incentives do delete an existing link and add a new link. There is no incentive to delete a link since $\tau \geq (1-\mu)\frac{1}{(n-1)^2}$.

To check if there is incentive to add a new link, the binding case is the link between i and $j = i+1$ such that $d_i(g) = 0$, $d_j(g) = 1$ and $d_{j+1} = 2$. That is, agent i has no links in g and agent j in a component of 3 contiguous agents.

Calculations shows that if there is no incentive to add the link ij if $\tau < \frac{4(1-\mu)(1-\mu^2)}{(1+2\mu)^2}\frac{1}{(n-1)^2}$.

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